

Strong convergence analysis of the stochastic exponential Rosenbrock scheme for the finite element discretization of semilinear SPDEs driven by multiplicative and additive noise

Jean Daniel Mukam, Antoine Tambue

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Abstract In this paper, we consider the numerical approximation of a general second order semilinear stochastic partial differential equation (SPDE) driven by multiplicative and additive noise. Our main interest is on such SPDEs where the nonlinear part is stronger than the linear part also called stochastic reactive dominated transport equations. Most numerical techniques, including current stochastic exponential integrators lose their good stability properties on such equations. Using finite element for space discretization, we propose a new scheme appropriated on such equations, called stochastic exponential Rosenbrock scheme (SERS) based on local linearization at every time step of the semi-discrete equation obtained after space discretization. We consider noise that is in a trace class and give a strong convergence proof of the new scheme toward the exact solution in the root-mean-square L^2 norm. Numerical experiments to sustain theoretical results are provided.

Keywords Exponential Rosenbrock-Euler method · Stochastic partial differential equations · Multiplicative & additive noise · Strong convergence · Finite element method · Errors estimate · Stochastic reactive dominated transport equations.

J. D. Mukam

Technische Universität Chemnitz, 09126 Chemnitz, Germany.
Tel.: +49-15213471370
E-mail: jean.d.mukam@aims-senegal.org
jean-daniel.mukam@s2015.tu-chemnitz.de

A. Tambue (Corresponding author)
The African Institute for Mathematical Sciences(AIMS) of South Africa and Stellenbosh University,
Center for Research in Computational and Applied Mechanics (CERECAM), and Department of Mathematics and Applied Mathematics, University of Cape Town, 7701 Rondebosch, South Africa.
Tel.: +27-785580321
E-mail: antonio@aims.ac.za, tambuea@gmail.com

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1 Introduction

The strong numerical approximation of an Itô stochastic partial differential equation defined in the bounded domain $\Lambda \subset \mathbb{R}^d$ ($d = 1, 2, 3$) is analyzed. The domain Λ is assumed to be a convex polygon, or has smooth boundary. Boundary conditions on the domain Λ are typically Neumann, Dirichlet or Robin conditions. More precisely, we consider in the abstract setting the following stochastic partial differential equation

$$dX(t) = [AX(t) + F(X(t))]dt + B(X(t))dW(t), \quad X(0) = X_0, \quad t \in [0, T] \quad (1)$$

on $H = L^2(\Lambda)$, $T > 0$ is a final time, F and B are nonlinear functions, X_0 is the initial data which may be random, A is a linear operator, unbounded, not necessarily self adjoint, and the generator of an analytic semigroup $S(t) := e^{tA}$, $t \geq 0$. The noise $W(t) = W(x, t)$ is a Q -Wiener process defined in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$. The filtration is assumed to fulfill the usual conditions (see [27, Definition 2.1.11]). We assume that the noise can be represented as

$$W(x, t) = \sum_{i \in \mathbb{N}^d} \sqrt{q_i} e_i(x) \beta_i(t), \quad t \in [0, T], \quad (2)$$

where q_i , e_i , $i \in \mathbb{N}^d$ are respectively the eigenvalues and the eigenfunctions of the covariance operator Q , and β_i are independent and identically distributed standard Brownian motion. Precise assumptions on F , B , X_0 and A will be given in the next section to ensure the existence of the unique mild solution X of (1) which has the following representation (see [25, 27])

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dW(s), \quad t \in (0, T].$$

In few cases, exact solutions are explicitly available, so numerical techniques are the only tools to provide good approximations in more general cases (see for examples [14, 18, 20, 22, 26, 37–39]). Approximations are done at two levels, spatial approximation and temporal approximation. For the spatial approximation, the finite difference, the finite element method and the Galerkin spectral method are usually used [14, 20, 22, 31, 38, 39]. As for PDEs, standard explicit time stepping methods for SPDEs are usually unstable for stiff problems and therefore severe time step constraint is needed. To overcome that drawback, standard implicit Euler methods are usually used [19, 20, 26]. Although standard implicit Euler methods¹ are stable, their implementation requires significantly more computational effort, specially full implicit methods, as Newton method is usually used to solve nonlinear algebraic equations

¹ Full implicit or semi-implicit methods

at each time step. Recently, stochastic exponential integrators [14, 22, 37] were appeared as non standard explicit methods efficient for SPDE (1). All stochastic exponential integrators analyzed in the literature for SPDEs [14, 22, 37] are bounded on the nonlinear problem as in (1) where the linear part A and the nonlinear function F are explicitly known a priori. Such approach is justified in situations where the nonlinear function F is small. Indeed when F is small the linear operator A drives the SPDE (1) and the good stability of the exponential integrators and semi-implicit method are ensured. In fact, in more realistic applications the nonlinear function F can be stronger. Typical examples are semilinear advection diffusion reaction equations with stiff reaction term. In such cases, the SPDE (1) is driven by the nonlinear operator F and both exponential integrators [14, 22, 37] and semi-implicit Euler [26] will behave as explicit Euler-Maruyama scheme, therefore their good stability properties are lost. To overcome this issue we propose in this work a new scheme called Stochastic Exponential Rosenbrock Scheme (SERS). Coupled with finite element for space discretization, the new scheme is based on a local linearization of the drift term at each time step in the corresponding semi-discrete problem of (1). The local linearization therefore weakens the nonlinear part of the drift such that the linearized semi-discrete problem is driven by its linear part, which changes at each time step. The standard stochastic exponential scheme [22] is applied at the end to that linearized semi-discrete problem and the corresponding scheme is our new scheme. The challenge here is to deal with the new discrete semigroup which indeed is a semigroup process, called stochastic perturbed semigroup. The key idea comes from the deterministic exponential Rosenbrock method [10–12, 23, 28]. Note that similar schemes for stochastic differential equation in finite dimension have been proposed in [2, 3]. Using some deterministic tools from [23], we propose a strong convergence proof of the new schemes where the linear operator A is not necessarily self adjoint. Note that the orders of convergence are the same with stochastic exponential schemes proposed in [22]. The deterministic part of this scheme is order 2 in time and has been proven to be efficient and robust in comparison to standard schemes in many applications [7, 34] where the perturbed semigroup and related matrix functions have been computed using the Krylov subspace technique [9] and fast Leja points technique [1, 34]. For our new stochastic scheme, numerical simulations show the good stability behavior of the new scheme compared with a stochastic exponential scheme proposed in [22], where the stochastic perturbed semigroup and related matrix functions are computed using Krylov subspace technique.

The rest of this paper is organized as follows. Section 2 is devoted to the mathematical setting, the numerical method and the main result. In Section 3 some preparatory results and the proof of the main result are provided. We end the paper in Section 4 with some numerical experiments to sustain our theoretical results.

2 Mathematical setting and main result

2.1 Main assumptions and well posedness

Before we state the well posedness result, let us define keys functional spaces, norms and notations that we will be use in the rest of the paper. Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|)$ be a separable Hilbert space. For all $p \geq 2$ and for a Banach space U , we denote by $L^p(\Omega, U)$ the Banach space of all equivalence classes of p integrable U -valued random variables. We denote by $L(U, H)$ the space of bounded linear mappings from U to H endowed with the usual operator norm $\|\cdot\|_{L(U, H)}$. By $\mathcal{L}_2(U, H) := HS(U, H)$, we denote the space of Hilbert Schmidt operators from U to H . For simplicity we use the notations $L(U, U) =: L(U)$ and $\mathcal{L}_2(U, U) =: \mathcal{L}_2(U)$. We assume that the covariance operator $Q : H \rightarrow H$ is positive and self-adjoint. Throughout this paper $W(t)$ is a Q -wiener process. The space of Hilbert Schmidt operator from $Q^{1/2}(H)$ to H is denoted by $L_2^0 := \mathcal{L}_2(Q^{1/2}(H), H) = HS(Q^{1/2}(H), H)$ with the corresponding norm $\|\cdot\|_{L_2^0}$ defined by

$$\|l\|_{L_2^0} := \|lQ^{1/2}\|_{HS} = \left(\sum_{i=1}^{\infty} \|lQ^{1/2}e_i\|^2 \right)^{1/2}, \quad l \in L_2^0,$$

where $(e_i)_{i=1}^{\infty}$ is an orthonormal basis of H . This definition is independent of the orthonormal basis of H .

In order to ensure the existence and the uniqueness of solution of (1) we make the following assumptions.

Assumption 1 [Linear operator A] $A : \mathcal{D}(A) \subset H \rightarrow H$ is a negative generator of an analytic semigroup $S(t) := e^{At}$.

Assumption 2 [Initial value X_0] We assume that $X_0 \in L^2(\Omega, \mathcal{D}((-A)^{\beta/2}))$, $0 < \beta < 2$.

As in the current literature on deterministic exponential Rosenbrock-Type methods [10, 11, 23, 29, 30], we make the following assumption on the nonlinear term.

Assumption 3 [Nonlinear term F] We assume that the nonlinear mapping $F : H \rightarrow H$ is Lipschitz continuous and Fréchet derivable with its derivative uniformly bounded, i.e. there exists a constant $C > 0$ such that

$$\|F(Y) - F(Z)\| \leq C\|Y - Z\|, \quad \text{and} \quad \|F'(v)\|_{L(H)} \leq C, \quad \forall v, Y, Z \in H.$$

As a consequence, there exists a constant $L > 0$ such that

$$\|F(Z)\| \leq \|F(0)\| + \|F(Z) - F(0)\| \leq \|F(0)\| + C\|Z\| \leq L(1 + \|Z\|),$$

for all $Z \in H$.

Following [25, Chapter 7] or [13, 19, 22, 38] we make the following assumption on the diffusion term.

Assumption 4 [Diffusion term] We assume that the operator $B : H \rightarrow L_2^0$ satisfies the global Lipschitz condition, i.e. there exists a positive constant C such that

$$\|B(Y) - B(Z)\|_{L_2^0} \leq C\|Y - Z\|, \quad \forall Y, Z \in H.$$

As a consequence, there exists a positive constant $L > 0$ such that

$$\|B(Z)\|_{L_2^0} \leq \|B(0)\|_{L_2^0} + \|B(Z) - B(0)\|_{L_2^0} \leq \|B(0)\|_{L_2^0} + C\|Z\| \leq L(1 + \|Z\|),$$

for all $Z \in H$. To establish our L^2 strong convergence result, we will also need the following further assumption on the diffusion term when $\beta \in [1, 2)$, which was also used in [13, 19, 20, 22].

Assumption 5 We assume that there exist two positive constants $c > 0$, and $\gamma \in (0, \frac{\beta}{10}]$ very small enough such that $B(\mathcal{D}(-A)^{\gamma/2}) \subset HS(Q^{1/2}(H), \mathcal{D}(-A)^{\gamma/2})$ and $\|(-A)^{\gamma/2}B(v)\|_{L_2^0} \leq c(1 + \|v\|_\gamma)$ for all $v \in \mathcal{D}((-A)^{\gamma/2})$, where β is the parameter defined in Assumption 2.

We equip $V_\alpha := \mathcal{D}((-A)^\alpha)$, $\alpha \in \mathbb{R}$ with the norm $\|v\|_\alpha := \|(-A)^{\alpha/2}v\|$, for all $v \in H$. It is well known that $(V_\alpha, \|\cdot\|_\alpha)$ is a Banach space [8]. Let us recall in the following proposition some semigroup properties of the operator $S(t)$ generated by A ² that will be useful in the rest of the paper.

Proposition 1 [Smoothing properties of the semigroup] [8] Let $\alpha > 0$, $\delta \geq 0$ and $0 \leq \gamma \leq 1$, then there exists a constant $C > 0$ such that

$$\begin{aligned} \|(-A)^\delta S(t)\|_{L(H)} &\leq Ct^{-\delta}, \quad t > 0, \\ \|(-A)^{-\gamma}(\mathbf{I} - S(t))\|_{L(H)} &\leq Ct^\gamma, \quad t \geq 0 \\ (-A)^\delta S(t) &= S(t)(-A)^\delta, \quad \text{on } \mathcal{D}((-A)^\delta) \\ \|D_t^l S(t)v\|_\delta &\leq Ct^{-l-(\delta-\alpha)/2}\|v\|_\alpha, \quad t > 0, \quad v \in \mathcal{D}((-A)^\alpha), \end{aligned}$$

where $l = 0, 1$, and $D_t^l = \frac{d^l}{dt^l}$.

If $\delta \geq \gamma$ then $\mathcal{D}((-A)^\delta) \subset \mathcal{D}((-A)^\gamma)$.

Theorem 6 [Well posedness result] [25, Theorem 7.4]

Let Assumption 1, Assumption 3 and Assumption 4 be satisfied. If X_0 is a F_0 -measurable H valued random variable, then there exists a unique mild solution X of problem (1) taking the form (3) and satisfying the following

$$\mathbb{P} \left[\int_0^T \|X(s)\|^2 ds < \infty \right] = 1,$$

and for any $p \geq 2$ there exists a constant $C = C(p, T) > 0$ such that

$$\sup_{t \in [0, T]} \mathbb{E} \|X(t)\|^p \leq C(1 + \mathbb{E} \|X_0\|^p).$$

² The proposition indeed is general and provides some estimates for any semigroup and its generator.

Furthermore from [13, Theorem 1] or [22, Theorem 2.6] it holds that for all $\gamma \in [0, 1)$, for all $p \geq 2$ there exists a positive constant C such that for all $t \in [0, T]$ we have

$$(\mathbb{E}\|X(t)\|_\gamma^p)^{1/p} \leq C \left(1 + (\mathbb{E}\|X_0\|_\gamma^p)^{1/p}\right). \quad (3)$$

2.2 Finite element discretization

In the rest of this paper, to simplify the presentation, we assume that the linear operator A is a second order. More precisely, we assume that our SPDE (1) is a second order semilinear parabolic and take the form

$$\begin{aligned} dX(t, x) = & [\nabla \cdot (\mathbf{D} \nabla X(t, x)) - \mathbf{q} \cdot \nabla X(t, x) + f(x, X(t, x))]dt \\ & + b(x, X(t, x))dW(t, x), \quad x \in \Lambda, \quad t \in [0, T], \end{aligned} \quad (4)$$

where the functions $f : \Lambda \times \mathbb{R} \rightarrow \mathbb{R}$ and $b : \Lambda \times \mathbb{R} \rightarrow \mathbb{R}$ are continuously differentiable with globally bounded derivatives. In the abstract framework (1), the linear operator A takes the form

$$Au = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(D_{ij}(x) \frac{\partial u}{\partial x_j} \right) - \sum_{i=1}^d q_i(x) \frac{\partial u}{\partial x_i}, \quad (5)$$

$$\mathbf{D} = (D_{i,j})_{1 \leq i,j \leq d} \quad \mathbf{q} = (q_i)_{1 \leq i \leq d}. \quad (6)$$

where $D_{ij} \in L^\infty(\Lambda)$, $q_i \in L^\infty(\Lambda)$. We assume that there is a positive constant $c_1 > 0$ such that

$$\sum_{i,j=1}^d D_{ij}(x) \xi_i \xi_j \geq c_1 |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad x \in \overline{\Omega}.$$

The functions $F : H \rightarrow H$ and $B : H \rightarrow HS(Q^{1/2}(H), H)$ are defined by

$$(F(v))(x) = f(x, v(x)) \quad \text{and} \quad (B(v)u)(x) = b(x, v(x)) \cdot u(x), \quad (7)$$

for all $x \in \Lambda$, $v \in H$, $u \in Q^{1/2}(H)$, with $H = L^2(\Lambda)$. For an appropriate eigenfunctions (e_i) such that $\sup_{i \in \mathbb{N}^d} \left[\sup_{x \in \Lambda} \|e_i(x)\| \right] < \infty$, it is well known [13, Section 4] that the nemystskii operator F related to f and the multiplication operator B associated to b defined in (7) satisfy Assumption 3 and Assumption 4 respectively. As in [6, 22], we introduce two spaces \mathbb{H} and V , such that $\mathbb{H} \subset V$; the two spaces depend on the boundary conditions of Λ and the domain of the operator A . For Dirichlet (or first-type) boundary conditions we take

$$V = \mathbb{H} = H_0^1(\Lambda) = \{v \in H^1(\Lambda) : v = 0 \quad \text{on} \quad \partial\Lambda\}.$$

For Robin (third-type) boundary condition and Neumann (second-type) boundary condition, which is a special case of Robin boundary condition, we take $V = H^1(\Lambda)$

$$\mathbb{H} = \{v \in H^2(\Lambda) : \partial v / \partial \mathbf{v}_\Lambda + \alpha_0 v = 0, \quad \text{on } \partial\Lambda\}, \quad \alpha_0 \in \mathbb{R},$$

where $\partial v / \partial \mathbf{v}_\Lambda$ is the normal derivative of v and \mathbf{v}_Λ is the exterior pointing normal $n = (n_i)$ to the boundary of Λ given by

$$\partial v / \partial \mathbf{v}_\Lambda = \sum_{i,j=1}^d n_i(x) D_{ij}(x) \frac{\partial v}{\partial x_j}, \quad x \in \partial\Lambda.$$

Using the Green's formula and the boundary conditions, the corresponding bilinear form associated to $-A$ is given by

$$a(u, v) = \int_\Lambda \left(\sum_{i,j=1}^d D_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^d q_i \frac{\partial u}{\partial x_i} v \right) dx, \quad u, v \in V,$$

for Dirichlet and Neumann boundary conditions, and

$$a(u, v) = \int_\Lambda \left(\sum_{i,j=1}^d D_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^d q_i \frac{\partial u}{\partial x_i} v \right) dx + \int_{\partial\Lambda} \alpha_0 u v dx, \quad u, v \in V.$$

for Robin boundary conditions. Using the Garding's inequality ([32]) we obtain

$$a(v, v) \geq \lambda_0 \|v\|_{H^1(\Lambda)}^2 - c_0 \|v\|^2, \quad \forall v \in V. \quad (8)$$

By adding and subtracting $c_0 X dt$ on the right hand side of (1), we have a new linear operator that we still call A corresponding to the new bilinear form that we still call a such that the following coercivity property holds

$$a(v, v) \geq \lambda_0 \|v\|_1^2, \quad v \in V. \quad (9)$$

Note that the expression of the nonlinear term F has changed as we included the term $-c_0 X$ in a new nonlinear term that we still denote by F . The coercivity property (9) implies that A is sectorial on $L^2(\Lambda)$ i.e. there exists C_1 , $\theta \in (\frac{1}{2}\pi, \pi)$ such that

$$\|(\lambda I - A)^{-1}\|_{L(L^2(\Lambda))} \leq \frac{C_1}{|\lambda|} \quad \lambda \in S_\theta, \quad (10)$$

where $S_\theta = \{\lambda \in \mathbb{C} : \lambda = \rho e^{i\phi}, \rho > 0, 0 \leq |\phi| \leq \theta\}$ (see [8]). Then A is the infinitesimal generator of a bounded analytic semigroup $S(t) := e^{tA}$ on $L^2(\Lambda)$ such that

$$S(t) := e^{tA} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{t\lambda} (\lambda I - A)^{-1} d\lambda, \quad t > 0, \quad (11)$$

where \mathcal{C} denotes a path that surrounds the spectrum of A . The coercivity property (9) also implies that $-A$ is a positive operator and its fractional powers are well defined for any $\alpha > 0$, by

$$\begin{cases} (-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{tA} dt, \\ (-A)^\alpha = ((-A)^{-\alpha})^{-1}, \end{cases} \quad (12)$$

where $\Gamma(\alpha)$ is the Gamma function (see [8]). Let's now turn to the discretization of our problem (1). We start by splitting the domain Λ in finite triangles. Let \mathcal{T}_h be the triangulation with maximal length h satisfying the usual regularity assumptions, and $V_h \subset V$ the space of continuous functions that are piecewise linear over the triangulation \mathcal{T}_h . We consider the projection P_h from $H = L^2(\Lambda)$ to V_h defined for every $u \in H$ by

$$\langle P_h u, \chi \rangle_H = \langle u, \chi \rangle_H, \quad \forall \chi \in V_h. \quad (13)$$

The discrete operator $A_h : V_h \rightarrow V_h$ is defined by

$$\langle A_h \phi, \chi \rangle_H = \langle A \phi, \chi \rangle_H = -a(\phi, \chi), \quad \forall \phi, \chi \in V_h, \quad (14)$$

Like A , A_h is also a generator of a semigroup $S_h(t) := e^{tA_h}$. As any semigroup and its generator, A_h and $S_h(t)$ satisfy the smoothing properties of Proposition 6 but with a uniform constant C , independent of h . Following [4, 6, 21, 35], we characterize the domain of the operator $(-A)^{k/2}$, $0 \leq k \leq 1$ as follow

$$\begin{aligned} \mathcal{D}((-A)^{k/2}) &= \mathbb{H} \cap H^k(\Lambda), \quad (\text{for Dirichlet boundary conditions}), \\ \mathcal{D}(-A) &= \mathbb{H}, \quad \mathcal{D}((-A)^{1/2}) = H^1(\Lambda), \quad (\text{for Robin boundary conditions}). \end{aligned}$$

The semi-discrete version of problem (1) consists to find $X^h(t) \in V_h$, $t \in (0, T]$ such that $X^h(0) = P_h X_0$ and

$$dX^h(t) = [A_h X^h(t) + P_h F(X^h(t))]dt + P_h B(X^h(t))dW(t), \quad t \in (0, T]. \quad (15)$$

We note that A_h and $P_h F$ satisfy the same assumptions as A and F respectively. We also note that $P_h B$ satisfies Assumption 4. Therefore, Theorem 6 ensures the existence of the unique mild solution $X^h(t)$ of (15) such that

$$\|X^h(t)\| \leq C(1 + \|P_h X_0\|) \leq C(1 + \|X_0\|), \quad \forall t \in [0, T]. \quad (16)$$

This mild solution of (15) is given by

$$\begin{aligned} X^h(t) &= S_h(t)X^h(0) + \int_0^t S_h(t-s)P_h F(X^h(s))ds \\ &\quad + \int_0^t S_h(t-s)P_h B(X^h(s))dW(s). \end{aligned} \quad (17)$$

The following lemma will be useful in our convergence analysis.

Lemma 1 *The following inequality holds*

$$\|(-A_h)^\alpha P_h X\| \leq C \|(-A)^\alpha X\|, \quad \forall 0 \leq \alpha \leq \frac{1}{2}, \quad \forall X \in \mathcal{D}((-A)^{2\alpha}).$$

Proof From the equivalence of norms (see [21, (3.12)]) we have

$$\|(-A_h)^{1/2} P_h v\| \leq C \|P_h v\|_{H^1(\Lambda)}, \quad v \in H^1(\Lambda). \quad (18)$$

Note that

$$\|P_h v\|_{H^1(\Lambda)}^2 = \|P_h v\|_{L^2(\Lambda)}^2 + \sum_{i=1}^d \left\| \frac{\partial(P_h v)}{\partial x_i} \right\|_{L^2(\Lambda)}^2, \quad (19)$$

where $\frac{\partial}{\partial x_i}$ stands for the weak derivative. Let $\mathcal{D}(\Lambda)$ be the set of functions $\varphi \in C^\infty(\Lambda)$ with compact support in Λ . Let $v \in L^2(\Lambda)$, for all $\varphi \in \mathcal{D}(\Lambda)$, we have

$$\left\langle \frac{\partial(P_h v)}{\partial x_i}, \varphi \right\rangle = - \left\langle P_h v, \frac{\partial \varphi}{\partial x_i} \right\rangle = - \left\langle v, P_h^* \frac{\partial \varphi}{\partial x_i} \right\rangle, \quad (20)$$

where $\langle \cdot, \cdot \rangle$ is a duality pairing between $\mathcal{D}'(\Lambda)$ and $\mathcal{D}(\Lambda)$, and $\frac{\partial \varphi}{\partial x_i}$ is the derivative of φ in the classical sense. From [23, Remark 2.1] we have

$$P_h^* \frac{\partial \varphi}{\partial x_i} = \frac{\partial(P_h^* \varphi)}{\partial x_i},$$

since P_h^* is a linear operator. So, the equality (20) yields

$$\left\langle \frac{\partial(P_h v)}{\partial x_i}, \varphi \right\rangle = - \left\langle v, \frac{\partial(P_h^* \varphi)}{\partial x_i} \right\rangle = \left\langle \frac{\partial v}{\partial x_i}, P_h^* \varphi \right\rangle = \left\langle P_h \frac{\partial v}{\partial x_i}, \varphi \right\rangle. \quad (21)$$

Since (21) holds for all $\varphi \in \mathcal{D}(\Lambda)$, it follows that $\frac{\partial(P_h v)}{\partial x_i} = P_h \frac{\partial v}{\partial x_i}$ in the weak sense. Inserting this latter relation in (19), using the fact that the projection P_h is bounded with respect to the norm $\|\cdot\|_{L^2(\Lambda)}$ and again the equivalence of norm [21, (3.12)] yields

$$\begin{aligned} \|P_h v\|_{H^1(\Lambda)}^2 &= \|P_h v\|_{L^2(\Lambda)}^2 + \sum_{i=1}^d \left\| P_h \frac{\partial v}{\partial x_i} \right\|_{L^2(\Lambda)}^2 \\ &= \|v\|_{L^2(\Lambda)}^2 + \sum_{i=1}^d \left\| \frac{\partial v}{\partial x_i} \right\|_{L^2(\Lambda)}^2 \\ &= \|v\|_{H^1(\Lambda)}^2 \leq C \|(-A)^{1/2} v\|. \end{aligned} \quad (22)$$

We therefore have

$$\|(-A_h)^{1/2} P_h v\| \leq C \|(-A)^{1/2} v\|. \quad (23)$$

Note that (23) remains true if we replace $\frac{1}{2}$ by 0. By interpolation theory we have

$$\|(-A_h)^\alpha P_h v\| \leq C \|(-A)^\alpha v\|, \quad \forall 0 \leq \alpha \leq \frac{1}{2}, \quad \forall v \in \mathcal{D}((-A)^{2\alpha}). \quad (24)$$

Let us recall the following well known lemma.

Lemma 2 [*Itô isometry*] [27, Proposition 2.3.5]

For any $t \in [0, T]$ and for any L_2^0 -valued predictable process $\phi(s)$, $s \in [0, t]$ the following equality holds

$$\mathbb{E} \left[\left\| \int_0^t \phi(s) dW(s) \right\|^2 \right] = \mathbb{E} \left[\int_0^t \|\phi(s)\|_{L_2^0}^2 ds \right].$$

The following two lemmas provide space and time regularity results of the mild solution of the semi-discrete problem (15). These lemmas play an important role in our convergence analysis. More results on the regularity of the mild solution of problem (1) can be found in [13, 20, 25].

Lemma 3 [*Space regularity of the mild solution $X^h(t)$*]

Let Assumption 1, Assumption 2, Assumption 3 and Assumption 4 be fulfilled with $\beta \in [0, 1)$, then for all $t \in [0, T]$, $X^h(t) \in L^2(\Omega, \mathcal{D}((-A)^{\beta/2}))$. Moreover, there exists a positive constant C independent of h such that

$$\|(-A_h)^{\beta/2} X^h(t)\|_{L^2(\Omega, H)} \leq C \left(1 + \|(-A)^{\beta/2} X_0\|_{L^2(\Omega, H)} \right). \quad (25)$$

Proof The proof follows the same lines as that of [22, Lemma 2.6] or [13, Theorem 1] or [20, Theorem 3.1] by making use of Lemma 1.

Lemma 4 [*Time regularity of the mild solution $X^h(t)$*] Let X^h be the mild solution of (15). If Assumption 1, Assumption 2, Assumption 3 and Assumption 4 are fulfilled with the corresponding $0 < \beta \leq 2$. For $0 < \beta < 1$, there exists a positive constant C independent of h such that for $t_1, t_2 \in [0, T]$, $t_1 < t_2$, we have

$$(\mathbb{E} \|X^h(t_2) - X^h(t_1)\|^2)^{1/2} \leq C(t_2 - t_1)^{\beta/2} (1 + (\mathbb{E} \|X_0\|_{\beta}^2)^{1/2}). \quad (26)$$

Moreover, if Assumption 5 is fulfilled with $1 \leq \beta \leq 2$, then there exists a positive constant C such that

$$(\mathbb{E} \|X^h(t_2) - X^h(t_1)\|^2)^{1/2} \leq C(t_2 - t_1)^{1/2} (1 + (\mathbb{E} \|X_0\|_{\beta}^2)^{1/2}). \quad (27)$$

Proof The proof follows the same lines as that of [22, Lemma 2.7] or [13, Theorem 1] or [20, Theorem 4.1] by making use of Lemma 1.

2.3 Fully discrete scheme

For the time discretization, we consider the one-step method which provides the numerical approximated solution X_m^h of $X^h(t_m)$ at discrete time $t_m = m\Delta t$, $m = 0, \dots, M$. The method is based on the continuous linearization of (15). More precisely we linearize (15) at each time step as

$$dX^h(t) = [A_h X^h(t) + J_m^h X^h(t) + G_m^h(X^h(t))]dt + P_h B(X^h(t))dW(t), \quad (28)$$

for all $t_m \leq t \leq t_{m+1}$, where J_m^h is the Fréchet derivative of $P_h F$ at X_m^h and G_m^h is the remainder at X_m^h . Both J_m^h and G_m^h are random functions and are defined for all $\omega \in \Omega$ by

$$J_m^h(\omega) := (P_h F)'(X_m^h(\omega)), \quad (29)$$

$$G_m^h(\omega)(X^h(t)) := P_h F(X^h(t)) - J_m^h(\omega)X^h(t). \quad (30)$$

Before build the new numerical scheme, let us recall the following important lemma.

Lemma 5 *For all $m \in \mathbb{N}$ and all $\omega \in \Omega$, the random linear operator $A_h + J_m^h(\omega)$ is the generator of a strongly continuous semigroup $S_m^h(\omega)(t) := e^{(A_h + J_m^h(\omega))t}$, uniformly bounded on $[0, T]$ called random (or stochastic) perturbed semigroup. Furthermore the following estimate holds*

$$\left\| e^{(A_h + J_m^h(\omega))\Delta t} e^{(A_h + J_{m-1}^h(\omega))\Delta t} \dots e^{(A_h + J_k^h(\omega))\Delta t} \right\|_{L(H)} \leq C, \quad \forall 0 \leq k \leq m,$$

where C is a positive constant independent of h , m , k , Δt and the sample ω .

Proof The proof follows the same lines as [23, Lemma 3.6] or [29, Lemma 4]. Here our Jacobian depends on $\omega \in \Omega$, the constant C is independent of the sample ω since there exists $L > 0$ such that $\|P_h F'(v)\|_{L(H)} \leq L\|F'(v)\|_{L(H)} < L$, $\forall v \in H = L^2(A)$, according to Assumption 3.

Given the solution $X^h(t_m)$ and the numerical solution X_m^h at t_m , we obtain from (28) the following mild representation form of $X^h(t_{m+1})$

$$\begin{aligned} X^h(t_{m+1}) &= e^{(A_h + J_m^h)\Delta t} X^h(t_m) + \int_{t_m}^{t_{m+1}} e^{(A_h + J_m^h)(t_{m+1}-s)} G_m^h(X^h(s)) ds \\ &\quad + \int_{t_m}^{t_{m+1}} e^{(A_h + J_m^h)(t_{m+1}-s)} P_h B(X^h(s)) dW(s). \end{aligned} \quad (31)$$

We note that (31) is the exact solution of (15) at t_{m+1} . To establish our numerical method we use the following approximations

$$G_m^h(X^h(t_m + s)) \approx G_m^h(X_m^h), \quad (32)$$

$$e^{(A_h + J_m^h)(t_{m+1}-s)} P_h B(X(s)) \approx e^{(A_h + J_m^h)\Delta t} P_h B(X_m^h). \quad (33)$$

Therefore the deterministic integral part of (31) can be approximated as follows

$$\begin{aligned} &\int_{t_m}^{t_{m+1}} e^{(A_h + J_m^h)(t_{m+1}-s)} G_m^h(X^h(s)) ds \\ &= \int_0^{\Delta t} e^{(A_h + J_m^h)(\Delta t-s)} G_m^h(X^h(t_m + s)) ds \\ &\approx G_m^h(X_m^h)(A_h + J_m^h)^{-1} (e^{(A_h + J_m^h)\Delta t} - \mathbf{I}). \end{aligned} \quad (34)$$

Inserting (34) in (31) and using the approximation $X^h(t_m) \approx X_m^h$ give the following approximation X_{m+1}^h of $X^h(t_{m+1})$, called Stochastic Exponential Rosenbrock Scheme (SERS)

$$\begin{aligned} X_{m+1}^h &= e^{(A_h + J_m^h)\Delta t} X_m^h + (A_h + J_m^h)^{-1} (e^{(A_h + J_m^h)\Delta t} - \mathbf{I}) G_m^h(X_m^h) \\ &\quad + e^{(A_h + J_m^h)\Delta t} P_h B(X_m^h) (W_{t_{m+1}} - W_{t_m}), \end{aligned} \quad (35)$$

with $X_0^h := X^h(0) = P_h X_0$. The numerical scheme (35) can be rewritten in the following equivalent form, which is efficient for implementation

$$X_{m+1}^h = X_m^h + \varphi_1(\Delta t(A_h + J_m^h)) [(A_h + J_m^h)(X_m^h + P_h B(X_m^h) \Delta W_m + G_m^h(X_m^h))],$$

where $\Delta W_m := W_{t_{m+1}} - W_{t_m} = \sqrt{\Delta t} \sum_{i=1}^{\infty} \sqrt{q_i} R_{i,m} e_i$, $R_{i,m}$ are independent, standard, normally distributed random variables with mean 0 and variance 1, and

$$\begin{aligned} \varphi_1(\Delta t(A_h + J_m^h)) &:= (A_h + J_m^h)^{-1} (e^{\Delta t(A_h + J_m^h)} - \mathbf{I}) \\ &= \int_0^{\Delta t} e^{(\Delta t-s)(A_h + J_m^h)} ds. \end{aligned}$$

We note that the operator $\varphi_1(\Delta t(A_h + J_m^h(\omega)))$ is uniformly bounded, i.e independently of h , m and ω (see e.g [10, Lemma 2.4]).

Remark 1 Note that the corresponding standard stochastic exponential scheme presented in [22] is given by

$$Z_{m+1}^h = Z_m^h + \varphi_1(\Delta t A_h) [A_h(Z_m^h + P_h B(Z_m^h) \Delta W_m + P_h F(Z_m^h))]. \quad (36)$$

This scheme will be called SETD1 and will be used in our numerical simulations for comparison with SERS scheme.

Remark 2 If the deterministic part is also approximated as the diffusion part (33), we will obtain the following new scheme

$$U_{m+1}^h = e^{(A_h + J_m^h)\Delta t} [U_m^h + P_h B(U_m^h) \Delta W_m + G_m^h(U_m^h)], \quad (37)$$

Our main result is also valid for scheme (37) and the extension of our proof to that scheme is done as in [22] without any issue.

Having the numerical method in hand, our goal is to analyze its strong convergence toward the exact solution in the root-mean-square L^2 sense. In the following subsection we state our strong convergence result, which is in fact our main result.

2.4 Main result

Throughout this paper we take $t_m = m\Delta t \in [0, T]$, where $T = M\Delta t$ for $m, M \in \mathbb{N}$, $m \leq M$, T is fixed, C is a generic constant that may change from one place to another. The main result of this paper is formulated in the following theorem.

Theorem 7 *Let $X(t_m)$ and X_m^h be respectively the mild solution (3) and the numerical approximation given by (35) at $t_m = m\Delta t$. Let Assumption 1, Assumption 2, Assumption 3 and Assumption 4 be fulfilled. For $0 < \beta < 1$, the following error estimate holds*

$$(\mathbb{E}\|X(t_m) - X_m^h\|^2)^{1/2} \leq C \left(h^\beta + \Delta t^{\beta/2} \right).$$

Moreover, under a strong regularity of the initial data, that is Assumption 2 and Assumption 5 are fulfilled with $\beta \in [1, 2)$, the following error estimate holds

$$(\mathbb{E}\|X(t_m) - X_m^h\|^2)^{1/2} \leq C \left(h^\beta + \Delta t^{1/2} \right).$$

As in [22, Remark 2.9], strong assumptions on the nonlinear functions F and B can allow to achieve a spatial error of order $\mathcal{O}(h^2)$.

Remark 3 For additive noise, smooth noise with further assumptions on the nonlinear term F should improve the time accuracy as in [16, 37].

Remark 4 Note that the semi-discrete problem (15) can be replaced by the following semi-discrete problem where the noise is truncated

$$dX^h(t) = [A_h X^h(t) + P_h F(X^h(t))]dt + P_h B(X^h(t))P_h dW(t), t \in [0, T]. \quad (38)$$

It was shown in [18] that in the case of additive noise with smooth covariance operator kernel, this truncation can be done severely without losing the spatial accuracy of the finite element method. Applying our stochastic exponential Rosenbrock scheme to (38) yields

$$\begin{aligned} Y_{m+1}^h &= e^{(A_h + J_m^h)\Delta t} Y_m^h + (A_h + J_m^h)^{-1} \left(e^{(A_h + J_m^h)\Delta t} - \mathbf{I} \right) G_m^h(Y_m^h) \\ &\quad + e^{(A_h + J_m^h)\Delta t} P_h B(Y_m^h) P_h (W_{t_{m+1}} - W_{t_m}). \end{aligned} \quad (39)$$

We note that Theorem 7 also holds for the numerical scheme (39). Parts of [35] can be used.

3 Proof of the main result

Before prove our main result, some preparatory results are needed.

3.1 Preparatory results

Lemma 6 *Let $(G_m^h(\omega))_m$ be defined by (30) satisfies the global Lipschitz condition with a uniform constant, i.e. there exists a positive constant $C > 0$, independent of h , m and ω such that*

$$\|G_m^h(\omega)(u^h) - G_m^h(\omega)(v^h)\| \leq C\|u^h - v^h\|, \quad \forall m \in \mathbb{N}, \quad \forall u^h, v^h \in V_h.$$

Proof Using Assumption 3 and relations (29)-(30), the proof is straightforward.

We introduce the Riesz representation operator $R_h : V \rightarrow V_h$ defined by

$$\langle AR_h v, \chi \rangle_H = \langle -Av, \chi \rangle_H = a(v, \chi), \quad \forall v \in V, \quad \forall \chi \in V_h. \quad (40)$$

It is well known (see [21, 22]) that A and A_h are related by $A_h R_h = P_h A$. Under the regularity assumptions on the triangulation and in view of the V -ellipticity (9), it is well known (see [6]) that for all $r \in \{1, 2\}$ the following errors estimates hold

$$\|R_h v - v\| + h\|R_h v - v\|_{H^1(\Omega)} \leq Ch^r \|v\|_{H^r(\Omega)}, \quad v \in V \cap H^r(\Omega). \quad (41)$$

Let us consider the following deterministic linear problem : find $u \in V$ such that

$$\frac{du}{dt} = Au, \quad u(0) = v, \quad t \in (0, T]. \quad (42)$$

The corresponding semi-discrete problem in space consist to find $u_h \in V_h$ such that

$$\frac{du_h}{dt} = A_h u_h, \quad u_h(0) = P_h v, \quad t \in (0, T]. \quad (43)$$

Let's define the following operator

$$T_h(t) := S(t) - S_h(t)P_h = e^{At} - e^{A_h t}P_h, \quad (44)$$

so that $u(t) - u_h(t) = T_h(t)v$. The estimate (41) was used in [22] to prove the key part of the following lemma.

Lemma 7 *The following estimate holds*

$$\|T_h(t)v\| \leq Ch^r t^{-(r-\alpha)/2} \|v\|_\alpha, \quad r \in [0, 2], \quad \alpha \leq r, \quad t \in (0, T]. \quad (45)$$

Proof The proof of Lemma 7 for $r \in [1, 2]$ can be found in [22, Lemma 3.1]. Using the stability property of $S(t)$ and $S_h(t)$, and the fact that the projection P_h is bounded, it follows that

$$\|S(t)v - S_h(t)P_h v\| \leq C\|v\|. \quad (46)$$

Inequality (46) shows that (45) holds for $r = 0$. Interpolating between $r = 0$ and $r = 2$ completes the proof of Lemma 7.

Lemma 8 *Let Assumption 1, Assumption 2, Assumption 3 and Assumption 4 be fulfilled with $0 < \beta < 1$. Then the mild solutions $X(t)$ and $X^h(t)$ given respectively by (3) and (17) satisfy the following error estimate*

$$\|X(t) - X^h(t)\|_{L^2(\Omega, H)} \leq Ch^\beta, \quad t \in (0, T].$$

Further, if Assumption 5 is fulfilled with $1 \leq \beta < 2$. Then the following error estimate holds

$$\|X(t) - X^h(t)\|_{L^2(\Omega, H)} \leq Ch^\beta, \quad t \in (0, T].$$

Proof [35, Theorem 6.1]

Lemma 9 *Under Assumption 1, for all $\omega \in \Omega$ the stochastic perturbed semigroup $S_m^h(\omega)(t)$ satisfies the following stability property*

(i) *For $\gamma_1, \gamma_2 \leq 1$, such that $0 \leq \gamma_1 + \gamma_2 \leq 1$ we have*

$$\|(-A_h)^{-\gamma_1}(S_m^h(\omega)(t) - \mathbf{I})(-A_h)^{-\gamma_2}\|_{L(H)} \leq Ct^{\gamma_1 + \gamma_2}, \quad t \in (0, T].$$

(ii) *For $\gamma_1 \geq 0$, we have*

$$\|S_m^h(\omega)(t)(-A_h)^{\gamma_1}\|_{L(H)} \leq Ct^{-\gamma_1}, \quad t \in (0, T], \quad \gamma_1 \geq 0,$$

(iii) *For $\gamma_1 \geq 0$, such that $0 \leq \gamma_2 < 1$ and $\gamma_2 - \gamma_1 \geq 0$ we have*

$$\|(-A_h)^{-\gamma_1}S_m^h(\omega)(t)(-A_h)^{\gamma_2}\|_{L(H)} \leq Ct^{\gamma_1 - \gamma_2}, \quad t \in (0, T].$$

where C is a positive constant independent of h , m , Δt and the sample ω .

Proof We recall that the perturbed semigroup satisfies the following variation of parameters formula (see [5, Chapter 3, Corollary 1.7] or [24, Section 3.1, Page 77])

$$S_m^h(\omega)(t)v = S_h(t)v + \int_0^t S_h(t-s)J_m^h(\omega)S_m^h(\omega)(s)v ds, \quad (47)$$

for all $v \in H$ and all $t \geq 0$. Then it follows that

$$(S_m^h(\omega)(t) - \mathbf{I})v = (S_h(t) - \mathbf{I})v + \int_0^t S_h(t-s)J_m^h(\omega)S_m^h(\omega)(s)v ds. \quad (48)$$

It is obvious that $(-A_h)^{-\gamma_2}v \in H$ for all $v \in H$. Then, replacing v in (48) by $(-A_h)^{-\gamma_2}v$ and pre-multiplying both right hand side of (48) by $(-A_h)^{-\gamma_1}$ yields

$$\begin{aligned} & (-A_h)^{-\gamma_1}(S_m^h(\omega)(t) - \mathbf{I})(-A_h)^{-\gamma_2}v \\ &= (S_h(t) - \mathbf{I})(-A_h)^{-\gamma_2 - \gamma_1}v \\ &+ \int_0^t (-A_h)^{-\gamma_1}S_h(t-s)J_m^h(\omega)S_m^h(\omega)(s)(-A_h)^{-\gamma_2}v ds. \end{aligned} \quad (49)$$

Taking the norm in both sides of (49) and using Proposition 1, the fact that $(-A_h)^{-\gamma_2}$ and $J_m^h(\omega)$ are uniformly bounded, it follows that

$$\begin{aligned} \|(-A_h)^{-\gamma_1}(S_m^h(\omega)(t) - \mathbf{I})(-A_h)^{-\gamma_2}v\| &\leq Ct^{\gamma_2+\gamma_1}\|v\| + C \int_0^t \|v\| ds \\ &\leq Ct^{\gamma_2+\gamma_1}\|v\|. \end{aligned}$$

Using the definition of the norm $\|\cdot\|_{L(H)}$ gives the desired result for (i). To prove (ii), we multiply (47) by $(-A_h)^{\gamma_1}$ and obtain

$$\begin{aligned} S_m^h(\omega)(t)(-A_h)^{\gamma_1}v &= S_h(t)(-A_h)^{\gamma_1}v \\ &+ \int_0^t S_h(t-s)J_m^h(\omega)S_m^h(\omega)(s)(-A_h)^{\gamma_1}v ds, \end{aligned} \quad (50)$$

for all $v \in H$ and all $t \geq 0$. Taking the norm in both sides of (50) and using the stability property of $S_h(t)$, $S_m^h(\omega)(t)$ with the uniform boundedness of $J_m^h(\omega)$ gives

$$\begin{aligned} \|S_m^h(\omega)(t)(-A_h)^{\gamma_1}v\| &\leq Ct^{-\gamma_1}\|v\| \\ &+ C \int_0^t \|S_m^h(\omega)(s)(-A_h)^{\gamma_1}\|_{L(H)}\|v\| ds. \end{aligned} \quad (51)$$

From (51) it holds that

$$\|S_m^h(\omega)(t)(-A_h)^{\gamma_1}\|_{L(H)} \leq Ct^{-\gamma_1} + C \int_0^t \|S_m^h(\omega)(s)(-A_h)^{\gamma_1}\|_{L(H)} ds. \quad (52)$$

Applying the continuous Gronwall's lemma to (52) completes the proof of (ii). To prove (iii), we recall that the perturbed semigroup satisfies the following variation of parameter formula (see [5, Page 161])

$$S_m^h(\omega)(t)v = S_h(t)v + \int_0^t S_m^h(\omega)(s)J_m^h(\omega)S_h(t-s)v ds, \quad \forall v \in H. \quad (53)$$

Replacing v in (53) by $(-A_h)^{\gamma_2}v$ and pre-multiplying both left hand sides of (53) by $(-A_h)^{-\gamma_1}$ we obtain

$$\begin{aligned} (-A_h)^{-\gamma_1}S_m^h(\omega)(t)(-A_h)^{\gamma_2}v &= S_h(t)(-A_h)^{\gamma_2-\gamma_1}v \\ &+ \int_0^t (-A_h)^{-\gamma_1}S_m^h(\omega)(s)J_m^h(\omega)S_h(t-s)(-A_h)^{\gamma_2}v ds. \end{aligned} \quad (54)$$

Taking the norm in both sides of (54) and using the stability property of the semigroup $S_h(t)$ yields

$$\begin{aligned} &\|(-A_h)^{-\gamma_1}S_m^h(\omega)(t)(-A_h)^{\gamma_2}v\| \\ &\leq Ct^{\gamma_1-\gamma_2}\|v\| + C \int_0^t \|(-A_h)^{-\gamma_1}\|_{L(H)}\|S_m^h(\omega)(s)\|_{L(H)}(t-s)^{-\gamma_2}\|v\| ds \\ &\leq Ct^{\gamma_1-\gamma_2}\|v\| + C \int_0^t (t-s)^{-\gamma_2}\|v\| ds \\ &\leq Ct^{\gamma_1-\gamma_2}\|v\| + Ct^{1-\gamma_2}\|v\| \leq Ct^{\gamma_1-\gamma_2}\|v\| + C\|v\| \leq Ct^{\gamma_1-\gamma_2}\|v\|. \end{aligned} \quad (55)$$

This ends the proof of (iii).

The following lemma is similar to [29, Lemma 4], but its proof is easier than that of [29, Lemma 6] since we don't use any further lemmas in its proof.

Lemma 10 *Under Assumption 1 and Assumption 3, the perturbed semigroup S_m^h satisfies the following stability property*

$$\left\| e^{(A_h + J_m^h(\omega))\Delta t} \dots e^{(A_h + J_k^h(\omega))\Delta t} (-A_h)^\nu \right\|_{L(H)} \leq C t_{m+1-k}^{-\nu}, \quad 0 \leq \nu < 1,$$

where C is a positive constant independent of $m, k, h, \Delta t$ and the sample ω .

Proof As in [23] we set

$$\begin{cases} S_{m,k}^h(\omega) := e^{(A_h + J_m^h(\omega))\Delta t} \dots e^{(A_h + J_k^h(\omega))\Delta t}, & \text{if } m \geq k \\ S_{m,k}^h(\omega) := \mathbf{I}, & \text{if } m < k \end{cases}$$

Using the telescopic sum, we can rewrite the perturbed semigroup $S_{m,k}^h(\omega)$ as follow

$$\begin{aligned} S_{m,k}^h(\omega) &= e^{A_h(t_{m+1}-k)} + e^{A_h(t_{m+1}-t_{k+1})} \left(e^{(A_h + J_k^h(\omega))\Delta t} - e^{A_h \Delta t} \right) \\ &\quad + \sum_{j=k+1}^m e^{A_h(t_{m+1}-t_{j+1})} \left(e^{(A_h + J_j^h(\omega))\Delta t} - e^{A_h \Delta t} \right) S_{j-1,k}^h(\omega). \end{aligned} \quad (56)$$

Multiplying both sides of (56) by $(-A_h)^\nu$ yields

$$\begin{aligned} &S_{m,k}^h(\omega)(-A_h)^\nu \\ &= e^{A_h t_{m+1}-k} (-A_h)^\nu + e^{A_h(t_{m+1}-t_{k+1})} \left(e^{(A_h + J_k^h(\omega))\Delta t} - e^{A_h \Delta t} \right) (-A_h)^\nu \\ &\quad + \sum_{j=k+1}^m e^{A_h(t_{m+1}-t_{j+1})} \left(e^{(A_h + J_j^h(\omega))\Delta t} - e^{A_h \Delta t} \right) S_{j-1,k}^h(\omega)(-A_h)^\nu. \end{aligned} \quad (57)$$

Taking the norm in both sides of (57) and using the stability property of $S_h(t)$ together with the inequality $\left\| e^{(A_h + J_m^h(\omega))\Delta t} - e^{A_h \Delta t} \right\|_{L(H)} \leq C \Delta t$ (see [23, (55)]) gives

$$\begin{aligned} &\left\| S_{m,k}^h(\omega)(-A_h)^\nu \right\|_{L(H)} \\ &\leq C t_{m+1-k}^{-\nu} + \left\| e^{A_h(t_{m+1}-t_{k+1})} \right\|_{L(H)} \left\| \left(e^{(A_h + J_k^h(\omega))\Delta t} - e^{A_h \Delta t} \right) (-A_h)^\nu \right\|_{L(H)} \\ &\quad + \sum_{j=k+1}^m \left\| e^{A_h(t_{m+1}-t_{j+1})} \right\|_{L(H)} \left\| e^{(A_h + J_j^h(\omega))\Delta t} - e^{A_h \Delta t} \right\|_{L(H)} \left\| S_{j-1,k}^h(\omega)(-A_h)^\nu \right\|_{L(H)} \\ &\leq C t_{m+1-k}^{-\nu} + C \left\| \left(e^{(A_h + J_k^h(\omega))\Delta t} - e^{A_h \Delta t} \right) (-A_h)^\nu \right\|_{L(H)} \\ &\quad + C \Delta t \sum_{j=k+1}^m \left\| S_{j-1,k}^h(\omega)(-A_h)^\nu \right\|_{L(H)}. \end{aligned} \quad (58)$$

From the variation of parameter formula (see [5, Corollary 1.7, Chapter 3]) we have

$$e^{(A_h + J_m^h(\omega))\Delta t} - e^{A_h \Delta t} = \int_0^{\Delta t} e^{A_h(\Delta t-s)} J_m^h(\omega) e^{(A_h + J_m^h(\omega))s} ds. \quad (59)$$

Multiplying both side of (59) by $(-A_h)^\nu$ gives

$$\begin{aligned} & \left(e^{(A_h + J_m^h(\omega))\Delta t} - e^{A_h \Delta t} \right) (-A_h)^\nu \\ &= \int_0^{\Delta t} e^{A_h(\Delta t-s)} J_m^h(\omega) e^{(A_h + J_m^h(\omega))s} (-A_h)^\nu ds. \end{aligned} \quad (60)$$

Taking the norm in both sides of (60), using the stability property of $e^{A_h t}$, the uniformly boundness of J_m^h and Lemma 9 (ii) with $\gamma_1 = \nu$ gives

$$\begin{aligned} & \left\| \left(e^{(A_h + J_m^h(\omega))\Delta t} - e^{A_h \Delta t} \right) (-A_h)^\nu \right\|_{L(H)} \\ & \leq \int_0^{\Delta t} \|e^{A_h(\Delta t-s)}\|_{L(H)} \|J_m^h(\omega)\|_{L(H)} \|e^{(A_h + J_m^h(\omega))s} (-A_h)^\nu\|_{L(H)} ds \\ & \leq C \int_0^{\Delta t} s^{-\nu} ds \leq C \Delta t^{1-\nu} = C t_1^{-\nu} \Delta t. \end{aligned} \quad (61)$$

Substituting (61) in (58) yields

$$\begin{aligned} & \|S_{m,k}^h(\omega)(-A_h)^\nu\|_{L(H)} \\ & \leq C t_{m+1-k}^{-\nu} + C t_1^{-\nu} \Delta t \|S_{k-1,k}^h(\omega)\|_{L(H)} + C \Delta t \sum_{j=k+1}^m \|S_{j-1,k}^h(\omega)(-A_h)^\nu\|_{L(H)} \\ & \leq C t_{m+1-k}^{-\nu} + C t_1^{-\nu} \Delta t \|S_{k-1,k}^h(\omega)\|_{L(H)} \\ & \quad + C \Delta t \sum_{j=k+1}^m t_{j+1-k}^{-\nu} \|S_{j-1,k}^h(\omega)(-A_h)^\nu\|_{L(H)} \\ & \leq C t_{m+1-k}^{-\nu} + C \Delta t \sum_{j=k}^m t_{j+1-k}^{-\nu} \|S_{j-1,k}^h(\omega)(-A_h)^\nu\|_{L(H)}. \end{aligned} \quad (62)$$

Note that $C \Delta t \sum_{j=k}^m t_{j+1-k}^{-\nu} \leq C T^{1-\nu} < \infty$. Applying the discrete Gronwall's inequality to (62) completes the proof of Lemma 10.

Gathering our preparatory results, we are now ready to proof our main result in Theorem 7.

3.2 Proof of Theorem 7

Using the standard technique in the error analysis, we split the fully discrete error in two terms

$$\begin{aligned} \|X(t_m) - X_m^h\|_{L^2(\Omega, H)} &\leq \|X(t_m) - X^h(t_m)\|_{L^2(\Omega, H)} + \|X^h(t_m) - X_m^h\|_{L^2(\Omega, H)} \\ &=: err_0 + err_1. \end{aligned}$$

Note that the space error err_0 is estimated by Lemma 8. It remains to estimate the time error err_1 . Note also that in the case of additive noise the proof may be straightforward. We estimate the time error err_1 for both $0 < \beta < 1$ and $1 \leq \beta < 2$ separately in the following two subsections.

3.2.1 Estimate of the time error for $0 < \beta < 1$

We recall that the exact solution at t_m of the semidiscrete problem is given by

$$\begin{aligned} X^h(t_m) &= e^{(A_h + J_{m-1}^h)\Delta t} X^h(t_{m-1}) \\ &\quad + \int_{t_{m-1}}^{t_m} e^{(A_h + J_{m-1}^h)(t_m - s)} G_{m-1}^h(X^h(s)) ds \\ &\quad + \int_{t_{m-1}}^{t_m} e^{(A_h + J_{m-1}^h)(t_m - s)} P_h B(X^h(s)) dW(s). \end{aligned} \quad (63)$$

We also recall that the numerical solution at t_m given by (35) can be rewritten as

$$\begin{aligned} X_m^h &= e^{(A_h + J_{m-1}^h)\Delta t} X_{m-1}^h \\ &\quad + \int_{t_{m-1}}^{t_m} e^{(A_h + J_{m-1}^h)(t_m - s)} G_{m-1}^h(X_{m-1}^h) ds \\ &\quad + \int_{t_{m-1}}^{t_m} e^{(A_h + J_{m-1}^h)\Delta t} P_h B(X_{m-1}^h) dW(s). \end{aligned} \quad (64)$$

If $m = 1$ then it follows from (63) and (64) that

$$\begin{aligned} &\|X(t_1) - X_1^h\|_{L^2(\Omega, H)} \\ &\leq \left\| \int_0^{\Delta t} e^{(A_h + J_0^h)(\Delta t - s)} [G_0^h(X^h(s)) - G_0^h(X_0^h)] ds \right\|_{L^2(\Omega, H)} \\ &\quad + \left\| \int_0^{\Delta t} \left[e^{(A_h + J_0^h)(\Delta t - s)} P_h B(X^h(s)) - e^{(A_h + J_0^h)\Delta t} P_h B(X_0^h) \right] dW(s) \right\|_{L^2(\Omega, H)} \\ &=: I + II. \end{aligned} \quad (65)$$

Using Lemma 5 and Lemma 6 it holds that

$$\begin{aligned}
I &= \left\| \int_0^{\Delta t} e^{(A_h + J_0^h)(\Delta t - s)} [G_0^h(X^h(s)) - G_0^h(X_0^h)] ds \right\|_{L^2(\Omega, H)} \\
&\leq \int_0^{\Delta t} \left\| e^{(A_h + J_0^h)(\Delta t - s)} [G_0^h(X^h(s)) - G_0^h(X_0^h)] \right\|_{L^2(\Omega, H)} ds \\
&\leq \int_0^{\Delta t} \left(\mathbb{E} \left[\left\| e^{(A_h + J_0^h)(\Delta t - s)} \right\|_{L(H)}^2 \|G_0^h(X^h(s)) - G_0^h(X_0^h)\|^2 \right] \right)^{1/2} ds \\
&\leq C \int_0^{\Delta t} \|X^h(s) - X_0^h\|_{L^2(\Omega, H)} ds. \tag{66}
\end{aligned}$$

Using inequality (16) in (66) and taking in account the fact that $X_0^h = P_h X_0$, we obtain

$$\begin{aligned}
I &\leq C \int_0^{\Delta t} (\mathbb{E} \|X^h(s) - X_0^h\|^2)^{1/2} ds \leq C \int_0^{\Delta t} (\mathbb{E} [\|X^h(s)\|^2 + \|X_0^h\|^2])^{1/2} ds \\
&\leq C \int_0^{\Delta t} (\mathbb{E}(1 + \|X_0\|^2))^{1/2} ds \leq C \Delta t. \tag{67}
\end{aligned}$$

Using the Itô's isometry property, we have

$$\begin{aligned}
II &= \left\| \int_0^{\Delta t} \left[e^{(A_h + J_0^h)(\Delta t - s)} P_h B(X^h(s)) - e^{(A_h + J_0^h)\Delta t} P_h B(X_0^h) \right] dW(s) \right\|_{L^2(\Omega, H)} \\
&= \left(\mathbb{E} \left\| \int_0^{\Delta t} \left[e^{(A_h + J_0^h)(\Delta t - s)} P_h B(X^h(s)) - e^{(A_h + J_0^h)\Delta t} P_h B(X_0^h) \right] dW(s) \right\|^2 \right)^{1/2} \\
&= \left(\int_0^{\Delta t} \mathbb{E} \left\| e^{(A_h + J_0^h)(\Delta t - s)} P_h B(X^h(s)) - e^{(A_h + J_0^h)\Delta t} P_h B(X_0^h) \right\|_{L_2^0}^2 ds \right)^{1/2}. \tag{68}
\end{aligned}$$

Using triangle inequality, Lemma 5, Assumption 4, (16) and the fact that $(a + b)^2 \leq 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$, $\sqrt{u + v} \leq \sqrt{u} + \sqrt{v}$ for all positive real

numbers u and v , it follows from (68) that

$$\begin{aligned}
& II \\
& \leq \left(\int_0^{\Delta t} \mathbb{E} \left[\left\| e^{(A_h + J_0^h)(\Delta t - s)} P_h B(X^h(s)) \right\|_{L_2^0}^2 + \left\| e^{(A_h + J_0^h)\Delta t} P_h B(X_0^h) \right\|_{L_2^0}^2 \right] ds \right)^{1/2} \\
& \leq \left(\int_0^{\Delta t} \mathbb{E} \left[2 \left\| e^{(A_h + J_0^h)(\Delta t - s)} P_h B(X^h(s)) \right\|_{L_2^0}^2 + 2 \left\| e^{(A_h + J_0^h)\Delta t} P_h B(X_0^h) \right\|_{L_2^0}^2 \right] ds \right)^{1/2} \\
& \leq \left(\int_0^{\Delta t} \mathbb{E} \left[2 \left\| e^{(A_h + J_0^h)(\Delta t - s)} \right\|_{L(H)}^2 \left\| P_h B(X^h(s)) \right\|_{L_2^0}^2 + 2 \left\| e^{(A_h + J_0^h)\Delta t} \right\|_{L(H)}^2 \right. \right. \\
& \quad \left. \left. \left\| P_h B(X_0^h) \right\|_{L_2^0}^2 \right] ds \right)^{1/2} \\
& \leq C \left(\int_0^{\Delta t} (1 + \mathbb{E} \|X^h(s)\|^2) ds + \int_0^{\Delta t} (1 + \mathbb{E} \|X_0^h\|^2) ds \right)^{1/2} \\
& \leq C \Delta t^{1/2}.
\end{aligned} \tag{69}$$

Inserting (69) and (67) in (65) yields

$$\|X^h(t_1) - X_1^h\|_{L^2(\Omega, H)} \leq C \Delta t^{1/2}. \tag{70}$$

For $m \geq 2$, we iterate the exact solution (63) at t_m by substituting $X^h(t_j)$, $j = 1, 2, \dots, m-1$ in (63) by their mild forms

$$\begin{aligned}
& X^h(t_m) \\
& = e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_0^h)\Delta t} X^h(0) + \int_{t_{m-1}}^{t_m} e^{(A_h + J_{m-1}^h)(t_m - s)} G_{m-1}^h(X^h(s)) ds \\
& + \int_{t_{m-1}}^{t_m} e^{(A_h + J_{m-1}^h)(t_m - s)} P_h B(X^h(s)) dW(s) \\
& + \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1} - s)} G_{m-k-2}^h(X^h(s)) ds \\
& + \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1} - s)} P_h B(X^h(s)) dW(s).
\end{aligned} \tag{71}$$

Similarly, for $m \geq 2$, we iterate the numerical solution (64) at t_m by substituting X_j^h , $j = 1, 2, \dots, m-1$ only in the first term of (64) by their expressions

$$\begin{aligned}
& X_m^h \\
& = e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_0^h)\Delta t} X^h(0) + \int_{t_{m-1}}^{t_m} e^{(A_h + J_{m-1}^h)(t_m - s)} G_{m-1}^h(X_{m-1}^h) ds \\
& + \int_{t_{m-1}}^{t_m} e^{(A_h + J_{m-1}^h)\Delta t} P_h B(X_{m-1}^h) dW(s) \\
& + \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1} - s)} \\
& \quad G_{m-k-2}^h(X_{m-k-2}^h) ds \\
& + \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} e^{(A_h + J_{m-k-2}^h)\Delta t} P_h B(X_{m-k-2}^h) dW(s).
\end{aligned} \tag{72}$$

Therefore, it follows from (72) and (73) and the triangle inequality that

$$\frac{1}{4} \|X^h(t_m) - X_m^h\|_{L^2(\Omega, H)}^2 \leq III + IV + V + VI, \quad (74)$$

where

$$III = \left\| \int_{t_{m-1}}^{t_m} e^{(A_h + J_{m-1}^h)(t_m - s)} \left[G_{m-1}^h(X^h(s)) - G_{m-1}^h(X_{m-1}^h) \right] ds \right\|_{L^2(\Omega, H)}^2$$

$$IV = \left\| \int_{t_{m-1}}^{t_m} \left(e^{(A_h + J_{m-1}^h)(t_m - s)} P_h B(X^h(s)) - e^{(A_h + J_{m-1}^h)\Delta t} P_h B(X_{m-1}^h) \right) dW(s) \right\|_{L^2(\Omega, H)}^2,$$

and

$$V = \left\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1} - s)} \right.$$

$$\left. (G_{m-k-2}^h(X^h(s)) - G_{m-k-2}^h(X_{m-k-2}^h)) ds \right\|_{L^2(\Omega, H)}^2$$

$$VI = \left\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} \right.$$

$$\left. \left(e^{(A_h + J_{m-k-2}^h)(t_{m-k-1} - s)} P_h B(X^h(s)) - e^{(A_h + J_{m-k-1}^h)\Delta t} P_h B(X_{m-k-2}^h) \right) dW(s) \right\|_{L^2(\Omega, H)}^2.$$

Using Holder's inequality, the stability property of $S_m^h(t)$, Lemma 6, the triangle inequality and the fact that $(a + b)^2 \leq 2a^2 + 2b^2$ yields

$$III$$

$$\leq \left(\int_{t_{m-1}}^{t_m} \left\| e^{(A_h + J_{m-1}^h)(t_m - s)} (G_{m-1}^h(X^h(s)) - G_{m-1}^h(X_{m-1}^h)) \right\|_{L^2(\Omega, H)} ds \right)^2$$

$$\leq \left(\int_{t_{m-1}}^{t_m} \left(\mathbb{E} \left[\left\| e^{(A_h + J_{m-1}^h)(t_m - s)} \right\|_{L(H)}^2 \left\| (G_{m-1}^h(X^h(s)) - G_{m-1}^h(X_{m-1}^h)) \right\|^2 \right] \right)^{1/2} ds \right)^2$$

$$\leq C \left(\int_{t_{m-1}}^{t_m} \left(\mathbb{E} \left\| G_{m-1}^h(X^h(s)) - G_{m-1}^h(X_{m-1}^h) \right\|^2 \right)^{1/2} ds \right)^2$$

$$\leq C \left(\int_{t_{m-1}}^{t_m} \left(\mathbb{E} \|X^h(s) - X_{m-1}^h\|^2 \right)^{1/2} ds \right)^2 = C \left(\int_{t_{m-1}}^{t_m} \|X^h(s) - X_{m-1}^h\|_{L^2(\Omega, H)} ds \right)^2$$

$$\leq C \left(\int_{t_{m-1}}^{t_m} \|X^h(s) - X^h(t_{m-1})\|_{L^2(\Omega, H)} ds \right)^2 + C \Delta t^2 \|X^h(t_{m-1}) - X_{m-1}^h\|_{L^2(\Omega, H)}^2. \quad (75)$$

Using Lemma 4, it follows from (75) that

$$III \leq C \left(\int_{t_{m-1}}^{t_m} (s - t_{m-1})^{\beta/2} \right)^2 + C \Delta t^2 \|X^h(t_{m-1}) - X_{m-1}^h\|_{L^2(\Omega, H)}^2$$

$$\leq C \Delta t^{2+\beta} + C \Delta t^2 \|X^h(t_{m-1}) - X_{m-1}^h\|_{L^2(\Omega, H)}^2. \quad (76)$$

The estimation of IV follows the same lines as the one of VI , let us sketch its estimation at the end.

We use inequality $(a + b)^2 \leq 2a^2 + 2b^2$ to split V into two terms and have

$$\begin{aligned}
V &\leq 2 \left\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h) \Delta t} \dots e^{(A_h + J_{m-k-1}^h) \Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \\
&\quad \left. (G_{m-k-2}^h(X^h(s)) - G_{m-k-2}^h(X^h(t_{m-k-2}))) ds \right\|_{L^2(\Omega, H)}^2 \\
&+ 2 \left\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h) \Delta t} \dots e^{(A_h + J_{m-k-1}^h) \Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \\
&\quad \left. (G_{m-k-2}^h(X^h(t_{m-k-2})) - G_{m-k-2}^h(X_{m-k-2}^h)) ds \right\|_{L^2(\Omega, H)}^2 \\
&=: 2V_1 + 2V_2.
\end{aligned} \tag{77}$$

Using triangle inequality gives

$$\begin{aligned}
V_1 &= \left\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h) \Delta t} \dots e^{(A_h + J_{m-k-1}^h) \Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \\
&\quad \left. (G_{m-k-2}^h(X^h(s)) - G_{m-k-2}^h(X^h(t_{m-k-2}))) ds \right\|_{L^2(\Omega, H)}^2 \\
&\leq m \sum_{k=0}^{m-2} \left\| \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h) \Delta t} \dots e^{(A_h + J_{m-k-1}^h) \Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \\
&\quad \left. (G_{m-k-2}^h(X^h(s)) - G_{m-k-2}^h(X_{m-k-2}^h)) ds \right\|_{L^2(\Omega, H)}^2 \\
&\leq m \sum_{k=0}^{m-2} \left(\int_{t_{m-k-2}}^{t_{m-k-1}} \left(\mathbb{E} \left\| e^{(A_h + J_{m-1}^h) \Delta t} \dots e^{(A_h + J_{m-k-1}^h) \Delta t} \right\|_{L(H)}^2 \right. \right. \\
&\quad \left. \left\| e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right\|_{L(H)}^2 \right. \\
&\quad \left. \left. \times \left\| (G_{m-k-2}^h(X^h(s)) - G_{m-k-2}^h(X^h(t_{m-k-2}))) \right\|^2 \right]^{1/2} ds \right)^2.
\end{aligned} \tag{78}$$

The smoothing properties of the semigroup combining with Lemma 5 and Lemma 6 yields

$$\begin{aligned}
V_1 &\leq Cm \sum_{k=0}^{m-2} \left(\int_{t_{m-k-2}}^{t_{m-k-1}} (\mathbb{E} \|G_{m-k-2}^h(X^h(s)) - G_{m-k-2}^h(X^h(t_{m-k-2}))\|^2)^{1/2} ds \right)^2 \\
&\leq Cm \sum_{k=0}^{m-2} \left(\int_{t_{m-k-2}}^{t_{m-k-1}} (\mathbb{E} \|X^h(s) - X^h(t_{m-k-2})\|^2)^{1/2} ds \right)^2 \\
&= Cm \sum_{k=0}^{m-2} \left(\int_{t_{m-k-2}}^{t_{m-k-1}} \|X^h(s) - X^h(t_{m-k-2})\|_{L^2(\Omega, H)} ds \right)^2.
\end{aligned} \tag{79}$$

Using Holder's inequality together with Lemma 4, it follows from (79) that

$$\begin{aligned} V_1 &\leq Cm\Delta t \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \|X^h(s) - X^h(t_{m-k-2})\|_{L^2(\Omega, H)}^2 ds \\ &\leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} (s - t_{m-k-2})^\beta ds \leq C\Delta t^\beta. \end{aligned} \quad (80)$$

Again using triangle inequality, the smoothing properties of the semigroup, Lemma 5, Lemma 6 and Holder's inequality yields

$$\begin{aligned} V_2 &= \left\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \\ &\quad \left. (G_{m-k-2}(X^h(t_{m-k-2})) - G_{m-k-2}(X_{m-k-2}^h)) ds \right\|_{L^2(\Omega, H)}^2 \\ &\leq m \sum_{k=0}^{m-2} \left\| \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-2}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \\ &\quad \left. (G_{m-k-2}(X^h(t_{m-k-2})) - G_{m-k-2}(X_{m-k-2}^h)) ds \right\|_{L^2(\Omega, H)}^2 \\ &\leq Cm\Delta t \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \|X^h(t_{m-k-2}) - X_{m-k-2}^h\|_{L^2(\Omega, H)}^2 ds \\ &\leq C \sum_{k=0}^{m-2} \Delta t \|X^h(t_{m-k-2}) - X_{m-k-2}^h\|_{L^2(\Omega, H)}^2 \\ &= C\Delta t \sum_{k=0}^{m-2} \|X^h(t_k) - X_k^h\|_{L^2(\Omega, H)}^2. \end{aligned} \quad (81)$$

Substituting (81) and (80) in (77) yields

$$V \leq C\Delta t^\beta + C \sum_{k=0}^{m-2} \Delta t \|X^h(t_k) - X_k^h\|_{L^2(\Omega, H)}^2. \quad (82)$$

To estimate IV , we use the triangle inequality to split in two parts

$$\begin{aligned} &VI \\ &\leq \left\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \\ &\quad \left. [P_h B(X^h(s)) - P_h B(X^h(t_{m-k-2}))] dW(s) \right\|_{L^2(\Omega, H)}^2 \\ &+ \left\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} \right. \\ &\quad \left. [e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} P_h B(X^h(t_{m-k-2})) - e^{(A_h + J_{m-k-2}^h)\Delta t} P_h B(X_{m-k-2}^h)] dW(s) \right\|_{L^2(\Omega, H)}^2 \\ &=: VI_1 + VI_2. \end{aligned} \quad (83)$$

Using the Itô isometry property, Lemma 5, Assumption 4 and the fact that $S_k^h(\omega)$ is uniformly bounded independently of h , k and the sample ω yields

$$\begin{aligned}
VI_1 &= \mathbb{E} \left\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \\
&\quad \left. [P_h B(X^h(s)) - P_h B(X^h(t_{m-k-2}))] dW(s) \right\|^2 \\
&= \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left\| e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \\
&\quad \left. P_h [B(X^h(s)) - B(X^h(t_{m-k-2}))] \right\|_{L_2^0}^2 ds \\
&\leq \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left[\left\| e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right\|_{L(H)}^2 \right. \\
&\quad \left. \times \left\| P_h [B(X^h(s)) - B(X^h(t_{m-k-2}))] \right\|_{L_2^0}^2 \right] ds \\
&\leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \|P_h B(X^h(s)) - P_h B(X^h(t_{m-k-2}))\|_{L_2^0}^2 ds \\
&\leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \|X^h(s) - X^h(t_{m-k-2})\|_{L^2(\Omega, H)}^2 ds. \tag{84}
\end{aligned}$$

Applying Lemma 4, it follows from (84) that

$$VI_1 \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} (s - t_{m-k-2})^\beta ds \leq C \Delta t^\beta. \tag{85}$$

Using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we split VI_2 in two terms

$$\begin{aligned}
&VI_2 \\
&= \left\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} \left[e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \right. \\
&\quad \left. \left. P_h B(X^h(t_{m-k-2})) - e^{(A_h + J_{m-k-2}^h)\Delta t} P_h B(X_{m-k-2}^h) \right] dW(s) \right\|_{L^2(\Omega, H)}^2 \\
&\leq 2 \left\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} \right. \\
&\quad \left. \left[e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} - e^{(A_h + J_{m-k-2}^h)\Delta t} \right] P_h B(X^h(t_{m-k-2})) dW(s) \right\|_{L^2(\Omega, H)}^2 \\
&\quad + 2 \left\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} e^{(A_h + J_{m-k-2}^h)\Delta t} \right. \\
&\quad \left. [P_h B(X^h(t_{m-k-2})) - P_h B(X_{m-k-2}^h)] dW(s) \right\|_{L^2(\Omega, H)}^2 \\
&=: 2VI_{21} + 2VI_{22}. \tag{86}
\end{aligned}$$

Using Itô isometry property and using the triangle inequality yields

$$\begin{aligned} VI_{21} &= \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left\| e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} \left[e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \right. \\ &\quad \left. \left. - e^{(A_h + J_{m-k-2}^h)\Delta t} \right] P_h B(X^h(t_{m-k-2})) \right\|_{L_2^0}^2 ds. \end{aligned} \quad (87)$$

As $S_k^h(t)$ is a semigroup, we obviously have

$$S_k^h(t+s) = S_k^h(t)S_k^h(s). \quad (88)$$

Using relation (88), Lemma 9 (i) with $\gamma_1 = \frac{\beta}{2}$ and $\gamma_2 = 0$ and Lemma 10 with $\nu = \frac{\beta}{2}$ in (87) allows to have

$$\begin{aligned} VI_{21} &= \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left\| e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} \right. \\ &\quad \left. (\mathbf{I} - S_{m-k-2}(s - t_{m-k-2})) S_{m-k-2}(t_{m-k-1} - s) P_h B(X^h(t_{m-k-2})) \right\|_{L_2^0}^2 ds \\ &\leq \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left[\left\| e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} (-A)^{\frac{\beta}{2}} \right\|_{L(H)}^2 \right. \\ &\quad \times \|(-A)^{-\frac{\beta}{2}} (\mathbf{I} - S_{m-k-2}(s - t_{m-k-2}))\|_{L(H)}^2 \|S_{m-k-2}(t_{m-k-1} - s)\|_{L(H)}^2 \\ &\quad \left. \times \|P_h B(X^h(t_{m-k-2}))\|_{L_2^0}^2 \right] ds \\ &\leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} t_{k+1}^{-\beta} (s - t_{m-k-2})^\beta \|S_{m-k-2}(t_{m-k-1} - s)\|_{L(H)}^2 \\ &\quad \|B(X^h(t_{m-k-2}))\|_{L_2^0}^2 ds. \end{aligned} \quad (89)$$

Using Assumption 4 and the fact that the random perturbed semigroup S_k^h is uniformly bounded independently of k, h and the sample ω , it follows from (89) that

$$\begin{aligned} VI_{21} &\leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} t_{k+1}^{-\beta} (s - t_{m-k-2})^\beta ds \\ &\leq C \Delta t^\beta \sum_{k=0}^{m-2} t_{k+1}^{-\beta} \Delta t \leq C \Delta t^\beta. \end{aligned} \quad (91)$$

Let us estimate VI_{22}

$$\begin{aligned} VI_{22} &:= \left\| \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} e^{(A_h + J_{m-k-2}^h)\Delta t} \right. \\ &\quad \left. [P_h B(X^h(t_{m-k-2})) - P_h B(X_{m-k-2}^h)] dW(s) \right\|_{L^2(\Omega, H)}^2. \end{aligned} \quad (92)$$

Indeed as in the estimate of VI_1 , the following estimate holds for VI_{22}

$$\begin{aligned}
VI_{22} &\leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \|X^h(t_{m-k-2}) - X_{m-k-2}^h\|_{L^2(\Omega, H)}^2 ds \\
&\leq C \sum_{k=0}^{m-2} \Delta t \|X^h(t_{m-k-2}) - X_{m-k-2}^h\|_{L^2(\Omega, H)}^2 \\
&= C \Delta t \sum_{k=0}^{m-2} \|X^h(t_k) - X_k^h\|_{L^2(\Omega, H)}^2.
\end{aligned} \tag{93}$$

Inserting (93) and (91) in (83) gives

$$VI_2 \leq C \Delta t^\beta + C \Delta t \sum_{k=0}^{m-2} \|X^h(t_k) - X_k^h\|_{L^2(\Omega, H)}^2. \tag{94}$$

Substituting estimates of VI_1 (85) and VI_2 (94) in (83) yields

$$VI \leq C \Delta t^\beta + C \Delta t \sum_{k=0}^{m-2} \|X^h(t_k) - X_k^h\|_{L^2(\Omega, H)}^2. \tag{95}$$

Following the same lines as for the estimate of VI , we obtain

$$\begin{aligned}
IV &= \left\| \int_{t_{m-1}}^{t_m} \left[e^{(A_h + J_{m-1}^h)(t_m-s)} P_h B(X^h(s)) \right. \right. \\
&\quad \left. \left. - e^{(A_h + J_{m-1}^h)\Delta t} P_h B(X_{m-1}^h) \right] dW(s) \right\|_{L^2(\Omega, H)}^2 \\
&\leq C \Delta t^\beta + C \Delta t \|X^h(t_{m-1}) - X_{m-1}^h\|_{L^2(\Omega, H)}^2.
\end{aligned} \tag{96}$$

Gathering estimates of III , IV , V and VI in (74) yields

$$\|X^h(t_m) - X_m^h\|_{L^2(\Omega, H)}^2 \leq C \Delta t^\beta + C \Delta t \sum_{k=0}^{m-1} \|X^h(t_k) - X_k^h\|_{L^2(\Omega, H)}^2. \tag{97}$$

Applying the discrete Gronwall's lemma to (97) yields

$$\|X^h(t_m) - X_m^h\|_{L^2(\Omega, H)} \leq C \Delta t^{\beta/2}. \tag{98}$$

Using Lemma 8 together with inequality (98) completes the proof of Theorem 7 for $0 < \beta < 1$.

3.2.2 Estimate of the time error for $1 \leq \beta < 2$

Note that the estimates of *III* and *V* in Subsection 3.2.1 hold for $\beta \in [1, 2)$. We only need to re-estimate *IV* and *VI*. We will only estimate *VI* in detail since they are term similar.

$$VI \leq VI_1 + VI_2, \quad (99)$$

where VI_1 and VI_2 are defined by (83) in Subsection 3.2.1. Applying Lemma 4, it follows from (84) that

$$VI_1 \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} (s - t_{m-k-2}) ds \leq C \Delta t. \quad (100)$$

Using the inequality $(a + b)^2 \leq 2a^2 + 2b^2$, we split VI_2 in two terms as

$$VI_2 \leq VI_{21} + VI_{22}, \quad (101)$$

where VI_{21} and VI_{22} are given by (86) in Subsection 3.2.1. We recall that from (93) the following estimate holds for VI_{22}

$$VI_{22} \leq C \Delta t \sum_{k=0}^{m-2} \|X^h(t_k) - X_k^h\|_{L^2(\Omega, H)}^2. \quad (102)$$

Using Itô isometry property and the triangle inequality, we split VI_{21} in two parts as

$$\begin{aligned} & VI_{21} \\ &= \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left\| e^{(A_h + J_{m-1}^h) \Delta t} \dots e^{(A_h + J_{m-k-1}^h) \Delta t} \right. \\ & \quad \left. \left[e^{(A_h + J_{m-k-2}^h)(t_{m-k-1} - s)} - e^{(A_h + J_{m-k-2}^h) \Delta t} \right] P_h B(X^h(t_{m-k-2})) \right\|_{L_2^0}^2 ds. \\ &\leq 2 \sum_{k=0}^{m-1} \left(\int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left\| e^{(A_h + J_{m-1}^h) \Delta t} \dots e^{(A_h + J_{m-k-1}^h) \Delta t} \left[e^{(A_h + J_{m-k-2}^h)(t_{m-k-1} - s)} \right. \right. \right. \\ & \quad \left. \left. \left. - e^{(A_h + J_{m-k-2}^h) \Delta t} \right] [P_h B(X^h(t_{m-k-2})) - P_h B(X(t_{m-k-2}))] \right\|_{L_2^0}^2 ds \right) \\ &+ 2 \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left\| e^{(A_h + J_{m-1}^h) \Delta t} \dots e^{(A_h + J_{m-k-1}^h) \Delta t} \left[e^{(A_h + J_{m-k-2}^h)(t_{m-k-1} - s)} \right. \right. \\ & \quad \left. \left. - e^{(A_h + J_{m-k-2}^h) \Delta t} \right] P_h B(X(t_{m-k-2})) \right\|_{L_2^0}^2 ds \\ &:= 2VI_{211} + 2VI_{212}. \end{aligned} \quad (103)$$

Using Lemma 5 yields

$$\begin{aligned}
& VI_{211} \\
& \leq C \sum_{k=0}^{m-2} \left(\int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left[\left\| e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} \right\|_{L(H)}^2 \right. \right. \\
& \times \left. \left\| e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} - e^{(A_h + J_{m-k-2}^h)\Delta t} \right\|_{L(H)}^2 \|X(t_{m-k-2}) - X^h(t_{m-k-2})\|^2 \right] ds \right) \\
& \quad (104)
\end{aligned}$$

Using Assumption 4 and Lemma 8, we have

$$\begin{aligned}
VI_{211} & \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \|X(t_{m-k-2}) - X^h(t_{m-k-2})\|^2 ds \\
& \leq Ch^{2\beta}. \quad (105)
\end{aligned}$$

Using Itô isometry property and inserting an appropriated power of $-A_h$ yields the following estimate

$$\begin{aligned}
& VI_{212} \\
& = \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left\| e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} \left[e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \right. \\
& \quad \left. \left. - e^{(A_h + J_{m-k-2}^h)\Delta t} \right] P_h B(X(t_{m-k-2})) \right\|_{L_2^0}^2 ds \\
& \leq \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left[\left\| e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} \right\|_{L(H)}^2 \left\| e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \right. \\
& \quad \left. \left. - e^{(A_h + J_{m-k-2}^h)\Delta t} \right\|_{L(H)}^2 \left(-A_h \right)^{-\frac{\gamma}{2}} \right\|_{L(H)}^2 \left(\left\| (-A_h)^{\frac{\gamma}{2}} P_h B(X(t_{m-k-2})) \right\|_{L_2^0}^2 \right) ds \\
& \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left[\left\| e^{(A_h + J_{m-1}^h)\Delta t} \dots e^{(A_h + J_{m-k-1}^h)\Delta t} (-A_h)^{\frac{1-\gamma}{2}} \right\|_{L(H)}^2 \right. \\
& \quad \left\| (-A_h)^{\frac{-1+\gamma}{2}} \left(e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} - e^{(A_h + J_{m-k-2}^h)\Delta t} \right) (-A_h)^{\frac{-\gamma}{2}} \right\|_{L(H)}^2 \\
& \quad \times \left\| (-A_h)^{\frac{\gamma}{2}} P_h B(X(t_{m-k-2})) \right\|_{L_2^0}^2 \left. \right] ds. \quad (106)
\end{aligned}$$

Using relation (88), Lemma 9 (i) with $\gamma_1 = \frac{1-\gamma}{2}$ and $\gamma_2 = \frac{\gamma}{2}$ in (106) gives

$$\begin{aligned}
& VI_{212} \\
& \leq C \sum_{k=0}^{m-2} \int_{t_{m-k-2}}^{t_{m-k-1}} t_{k+1}^{-1+\gamma} \mathbb{E} \left[\|(-A_h)^{\frac{1+\gamma}{2}} \left(e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} - e^{(A_h + J_{m-k-2}^h)\Delta t} \right) \right. \\
& \quad \times (-A_h)^{-\frac{\gamma}{2}} \|_{L(H)}^2 \|(-A_h)^{\frac{\gamma}{2}} P_h B(X(t_{m-k-2}))\|_{L_2^0}^2 \Big] ds \\
& \leq C \sum_{k=0}^{m-2} t_{k+1}^{-1+\gamma} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left[\left\| (-A_h)^{-\frac{1+\gamma}{2}} \left[e^{(A_h + J_{m-k-2}^h)(t_{m-k-1}-s)} \right. \right. \right. \\
& \quad \left. \left. \left. - e^{(A_h + J_{m-k-2}^h)\Delta t} \right] (-A_h)^{-\frac{\gamma}{2}} \right\|_{L(H)}^2 \|(-A_h)^{\frac{\gamma}{2}} P_h B(X(t_{m-k-2}))\|_{L_2^0}^2 \right] ds.
\end{aligned} \tag{107}$$

Using Lemma 9 (iii) with $\gamma_1 = \gamma_2 = \frac{1-\gamma}{2}$, and Lemma 10 with $\nu = \frac{1-\gamma}{2}$ yields

$$\begin{aligned}
VI_{212} & \leq C \sum_{k=0}^{m-2} t_{k+1}^{-1+\gamma} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left[\|(-A_h)^{-\frac{1+\gamma}{2}} S_{m-k-2}^h(t_{m-k-1}-s) \right. \\
& \quad \left. (S_{m-k-2}^h(s-t_{m-k-2}) - \mathbf{I})(-A_h)^{-\frac{\gamma}{2}} \|_{L(H)}^2 \|(-A_h)^{\frac{\gamma}{2}} P_h B(X(t_{m-k-2}))\|_{L_2^0}^2 \right] ds \\
& \leq C \sum_{k=0}^{m-2} \left(t_{k+1}^{-1+\gamma} \int_{t_{m-k-2}}^{t_{m-k-1}} \mathbb{E} \left[\left\| (-A_h)^{-\frac{1+\gamma}{2}} S_{m-k-2}^h(t_{m-k-1}-s)(-A_h)^{\frac{1-\gamma}{2}} \right\|_{L(H)}^2 \right. \right. \\
& \quad \times \left\| (-A_h)^{-\frac{1+\gamma}{2}} (S_{m-k-2}^h(s-t_{m-k-2}) - \mathbf{I})(-A_h)^{-\frac{\gamma}{2}} \right\|_{L(H)}^2 \\
& \quad \left. \left. \|(-A_h)^{\frac{\gamma}{2}} P_h B(X(t_{m-k-2}))\|_{L_2^0}^2 \right] ds \right) \\
& \leq C \sum_{k=0}^{m-2} t_{k+1}^{-1+\gamma} \int_{t_{m-k-2}}^{t_{m-k-1}} (s-t_{m-k-2}) \mathbb{E} \|(-A_h)^{\frac{\gamma}{2}} P_h B(X(t_{m-k-2}))\|_{L_2^0}^2 ds \\
& \leq C \Delta t^2 \sum_{k=0}^{m-2} t_{k+1}^{-1+\gamma} \mathbb{E} \|(-A_h)^{\frac{\gamma}{2}} P_h B(X(t_{m-k-2}))\|_{L_2^0}^2.
\end{aligned} \tag{108}$$

Using the definition of the L_2^0 norm, Assumption 5, Lemma 1, Theorem 6 and inequality (3), we have

$$\begin{aligned}
\mathbb{E} \|(-A_h)^{\frac{\gamma}{2}} P_h B(X(t_{m-k-2}))\|_{L_2^0}^2 & = \mathbb{E} \left[\sum_{i=0}^{\infty} \|(-A_h)^{\frac{\gamma}{2}} P_h B(X(t_{m-k-2})) Q^{1/2} e_i\|^2 \right] \\
& \leq C \mathbb{E} \left[\sum_{i=0}^{\infty} \|(-A)^{\frac{\gamma}{2}} B(X(t_{m-k-2})) Q^{1/2} e_i\|^2 \right] \\
& = C \mathbb{E} \|(-A)^{\frac{\gamma}{2}} B(X(t_{m-k-2}))\|_{L_2^0}^2 \\
& \leq C \mathbb{E} (1 + \|(-A)^{\frac{\gamma}{2}} X(t_{m-k-2})\|^2) \\
& \leq C(1 + \mathbb{E}(\|X_0\|_\gamma^2)) < \infty.
\end{aligned} \tag{109}$$

Substituting (109) in (108) yields

$$VI_{212} \leq C\Delta t^2 \sum_{k=0}^{m-2} t_{k+1}^{-1+\gamma} \leq C\Delta t. \quad (110)$$

Inserting (110) and (105) in (103) gives

$$VI_{21} \leq Ch^{2\beta} + C\Delta t. \quad (111)$$

Inserting (111) and (102) in (101) gives

$$VI_2 \leq Ch^{2\beta} + C\Delta t + C\Delta t \sum_{k=0}^{m-2} \|X^h(t_k) - X_k^h\|_{L^2(\Omega, H)}^2. \quad (112)$$

Substituting estimates of VI_2 (112) and VI_1 (100) in (99) yields

$$VI \leq Ch^{2\beta} + C\Delta t + C\Delta t \sum_{k=0}^{m-2} \|X^h(t_k) - X_k^h\|_{L^2(\Omega, H)}^2. \quad (113)$$

Following the same lines as for VI , we obtain

$$\begin{aligned} IV &= \left\| \int_{t_{m-1}}^{t_m} \left[e^{(A_h + J_{m-1}^h)(t_m-s)} P_h B(X^h(s)) \right. \right. \\ &\quad \left. \left. - e^{(A_h + J_{m-1}^h)\Delta t} P_h B(X_{m-1}^h) \right] dW(s) \right\|_{L^2(\Omega, H)}^2 \\ &\leq Ch^{2\beta} + C\Delta t + C\Delta t \|X^h(t_{m-1}) - X_{m-1}^h\|_{L^2(\Omega, H)}^2. \end{aligned} \quad (114)$$

Therefore gathering the current estimates of IV and VI , and the estimates of III and V from Subsection 3.2.1 in the inequality (74) yields

$$\begin{aligned} &\|X^h(t_m) - X_m^h\|_{L^2(\Omega, H)}^2 \\ &\leq Ch^{2\beta} + C\Delta t + C\Delta t \sum_{k=0}^{m-1} \|X^h(t_k) - X_k^h\|_{L^2(\Omega, H)}^2. \end{aligned} \quad (115)$$

Applying the discrete Gronwall's lemma to (115) yields

$$\|X^h(t_m) - X_m^h\|_{L^2(\Omega, H)} \leq Ch^\beta + C\Delta t^{1/2}. \quad (116)$$

4 Numerical simulations

Here we provide two examples to sustain our theoretical results. The first example has exact solution. The reference solution or "the exact solution" using in the errors computation for our second example is taken to be the numerical solution with small time step. In the legends of our graphs, we use the following notations

1. SERS denotes the strong errors from our SERS scheme.
2. SETD1 denotes the strong errors from the stochastic exponential scheme [22] given by (36).

The exponential matrix function φ_1 is computed by Krylov subspace technique with fixed dimension $m = 10$ and tolerance $tol = 10^{-6}$ [9, 32, 34]. Note that we compute at every time step the action on the exponential matrix function on a vector and not the whole exponential matrix function. Our code was implemented in Matlab 8.1.

4.1 Additive noise with exact solution

We first consider the following stochastic reaction diffusion equation with stiff reaction driven by additive noise in two dimensions with Neumann boundary conditions

$$dX(t) = [D\Delta X(t) - 100X(t)]dt + dW(t), \quad X(0) = X_0, \quad t \in [0, T], \quad (117)$$

on the domain $\Omega = [0, L_1] \times [0, L_2]$, $D = 10^{-1}$. A simple computation shows that the eigenfunctions $\{e_{i,j}\}_{i,j \geq 0} = \{e_i^{(1)} \otimes e_j^{(2)}\}_{i,j \geq 0}$ with the corresponding eigenvalues $\{\lambda_{i,j}\}_{i,j \geq 0} = \{(\lambda_i^{(1)})^2 + (\lambda_j^{(2)})^2\}$ of $-\Delta$ are given by

$$e_0^{(l)}(x) = \sqrt{\frac{1}{L_l}}, \quad e_i^{(l)}(x) = \sqrt{\frac{2}{L_l}} \cos(\lambda_i^{(l)} x), \quad \lambda_0^{(l)} = 0, \quad \lambda_i^{(l)} = \frac{i\pi}{L_l}, \quad (118)$$

where $l = 1, 2$, $x \in \Omega$ and $i \in \mathbb{N}$. In the abstract form (1) our linear operator A is taken to be $A = D\Delta$ and $F(X) = -100X$ which obviously satisfies Assumption 3. We take $L_1 = L_2 = 1$ and the triangulation \mathcal{T} has been constructed from uniform cartesian grid of sizes $\Delta x = \Delta y = 1/100$.

We assume that the covariance operator Q and A have the same eigenfunctions. We take the following values for $\{q_{i,j}\}_{i+j > 0}$ in the representation (2)

$$q_{i,j} = \frac{1}{(i^2 + j^2)^r}, \quad r > 0. \quad (119)$$

To have trace class noise, it is enough to consider $r > 1/2$. In this example $b(x, X) = 1$, therefore B defined in (7) obviously satisfies Assumption 4 and Assumption 5. The exact solution of (117) is constructed in [35]. Figure 1 shows the strong convergence of SERS and SETD1 schemes. This figure also shows that SETD1 is unstable for large time steps. We can observe the good stability property of the new SERS scheme even for large time steps. We can also observe that the two schemes have the same order of accuracy. Indeed although SETD1 seems to be more accurate, the two graphs become very close for small time step. The orders of convergence of the two methods are 0.4871 and 0.4880 for SERS and SETD1 schemes respectively, which are very close to theoretical results. Note that we only use the stable part of the data in the computation of the order of convergence of SETD1 scheme.

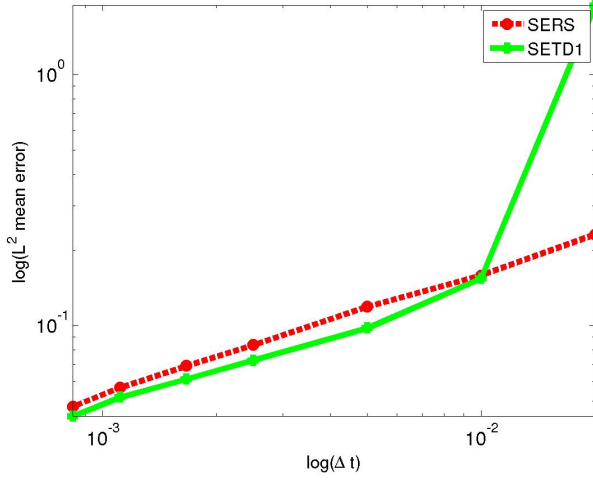


Fig. 1 Strong convergence of SERS and SETD1 scheme, we can also observe that SETD1 is unstable for large time steps. The orders of convergence of the two methods are 0.4871 and 0.4880 for SERS and SETD1 schemes respectively. The noise regularity parameter used is $r = 0.7$ and 50 samples are used in the errors computation.

4.2 Multiplication noise without exact solution

As a more challenging example we consider the the stochastic advection-diffusion-reaction SPDE with multiplicative noise in two dimensions on the domain $\Omega = [0, 1] \times [0, 1]$.

$$dX = \left[\nabla \cdot (\mathbf{D} \nabla X) - \nabla \cdot (\mathbf{q} X) - \frac{10X}{X+1} \right] dt + X dW. \quad (120)$$

$$\mathbf{D} = \begin{pmatrix} 10^{-2} & 0 \\ 0 & 10^{-3} \end{pmatrix} \quad (121)$$

with mixed Neumann-Dirichlet boundary conditions. The Dirichlet boundary condition is $X = 1$ at $x = 0$ and we use the homogeneous Neumann boundary conditions elsewhere. The Darcy velocity \mathbf{q} is obtained as in [22] and to deal with high Péclet flows we discretize in space using finite volume method (viewed as the finite element method as in [33]) in rectangular grid of sizes $\Delta x = \Delta y = 1/110$. The reference solution or "the exact solution" using in the errors computation is the numerical solution with the time step $\Delta t = 1/2048$. Relatively small time steps are used to stabilize the scheme SETD1. The noise used is the same as in the first example with (119). Our linear operator A is given by

$$A = \nabla \cdot \mathbf{D} \nabla(\cdot) - \nabla \cdot \mathbf{q}(\cdot). \quad (122)$$

and the functions f and b are given by

$$f(x, u) = \frac{-10u}{u+1}, \quad b(x, u) = u, \quad \forall x \in \Omega, \quad u \in \mathbb{R}. \quad (123)$$

Therefore, from [13, Section 4] it follows that the operators F and B defined by (7) fulfil obviously Assumption 3, Assumption Assumption 4 and Assumption 5.

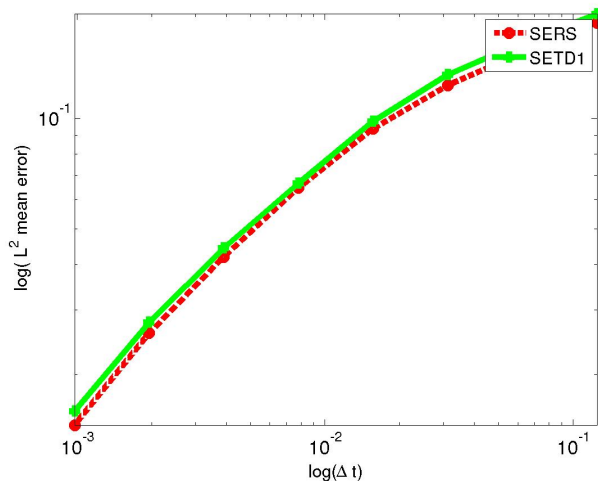


Fig. 2 Strong convergence of SERS and SETD1 scheme can be observed. The orders of convergence of the two methods are 0.5167 and 0.5137 for SERS and SETD1 schemes respectively. The noise regularity parameter used is $r = 0.6$ and 50 samples are used in the errors computation.

Figure 2 shows the strong convergence of SERS scheme and SETD1 scheme presented in [22]. We can also observe that the two schemes have the same order of accuracy. Indeed although SERS seems be more accurate, the difference between the two errors is small. The orders of convergence of the two methods are 0.5167 and 0.5137 for SERS and SETD1 schemes respectively, which are very close to theoretical results.

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